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NONLINEAR BEAM-BEAM RESONANCES IN DAΦNE

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Abstract

In this paper we calculate the parameters of the nonlinear resonances introduced by beam-beam interactions in the DAΦNE main rings.

Introduction

It is known that even in a linear lattice beam-beam collisions introduce nonlinear resonances due to the essentially nonlinear nature of the beam space charge forces:

$$l\nu_x + m\nu_y + n\nu_s = k \tag{1}$$

where ν_x and ν_y are the horizontal and vertical betatron tunes, ν_s is the synchrotron oscillation tune; k, l, m, n are integers, with m and $(l + n)$ being even numbers. Note that ν_x, ν_y, ν_s are the actual tunes taking into account tune shifts due to the beam-beam interaction.

The nonlinear resonances can set the most severe limits on the performance of a colliding machine. Through resonance streaming [1] and phase convection [2] a single resonance can blow-up the beam core reducing luminosity or lead to the growth of the bunch distribution tails affecting luminosity lifetime. Stochastic motion occurs when the resonances overlap.

In this paper we estimate the resonance parameters for the DAΦNE working point with $\nu_{x0} = 5.18$ and $\nu_{y0} = 6.15$ considering only one interaction point and assuming that the parasitic collisions have a negligible effect. We start first with two-dimensional betatron motion of a single particle interacting with a counter-rotating strong beam and find the strength of nonlinear resonances closely following the treatment of [3]. Then, the longitudinal bunch size and crossing angle are taken into account by introducing the so called "suppression factor" [4,5].

An isolated resonance

The Hamiltonian of two dimensional motion can be written in the following form:

$$H = \frac{K_x x^2}{2} + \frac{K_y y^2}{2} + \frac{x'^2}{2} + \frac{y'^2}{2} + V(x,y) \sum_{k=-\infty}^{\infty} \delta(z - 2\pi kR) \tag{2}$$

where K_x and K_y are the focusing functions; x, x' and y, y' are the betatron coordinates and canonical momenta normalized to the standard deviations σ_x, σ_x' and σ_y, σ_y'; 2πR is the length of the ideal trajectory; $V(x,y)$ is the beam- beam interaction potential.

After transformation of the Hamiltonian (2) to action-angle variables (x, x', y, y')$\rightarrow (I_x, \vartheta_x, I_y, \vartheta_y)$ and by expanding the potential V(x,y) in a Fourier series and averaging over rapidly oscillating phases the Hamiltonian of a single isolated resonance takes the form:

$$H_1 = \nu_x I_x + \nu_y I_y + V_{oo} + V_{lm} \cos(l\theta_x + m\theta_y - k\theta) \tag{3}$$

with

$$V_{00} = \frac{1}{(2\pi)^3} \int_0^{2\pi} \int_0^{2\pi} V(I_x, \theta_x, I_y, \theta_y) d\theta_x d\theta_y \tag{4}$$

and

$$V_{lm} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} V \cos l\theta_x \cos m\theta_y d\theta_x d\theta_y \tag{5}$$

According to the standard definitions [6] the horizontal and vertical tune shifts are

$$\Delta\nu_x = \frac{\partial V_{00}}{\partial I_x} \quad ; \quad \Delta\nu_y = \frac{\partial V_{00}}{\partial I_y} \tag{6}$$

and the resonance half-width in the action plane is:

$$\Delta I_x = 2l\sqrt{\left|\frac{V_{lm}}{\alpha}\right|}; \quad \Delta I_y = 2m\sqrt{\left|\frac{V_{lm}}{\alpha}\right|} \tag{7}$$

where the nonlinearity α is defined by [3]:

$$\alpha = m^2 \frac{\partial \Delta \nu_y}{\partial I_y} + 2ml \frac{\partial \Delta \nu_y}{\partial I_x} + l^2 \frac{\partial \Delta \nu_x}{\partial I_x} \tag{8}$$

For a very flat bunch with $\sigma_x >> \sigma_y$ (which is the case for DAΦNE) and the amplitudes of vertical oscillations satisfying the condition $|y\sigma_y| << \sigma_x$ the transverse beam-beam kicks can be represented in the following form [3]:

$$\Delta x' = -\xi_x f_x ; \quad \Delta y' = -\xi_y f_y \tag{9}$$

with f_x and f_y being dimensionless beam-beam forces:

$$f_x = 4\pi\sqrt{2} F_D\left(\frac{x}{\sqrt{2}}\right) \quad ; \quad f_y = (2\pi)^{3/2} e^{-x^2/2} \Phi\left(\frac{y}{\sqrt{2}}\right) \tag{10}$$

where $F_D(z)$ is the Dawson function and $\Phi(z)$ the error function. $\varsigma_x ; \ \varsigma_y$ are linear tune shifts parameters.

This allows to rewrite the expressions (6) for the tune shifts and (5) for V_{lm} as:

$$\begin{aligned} \Delta \nu_x &= \xi_x \frac{1}{(2\pi)^3 A_x} \int_0^{2\pi} \int_0^{2\pi} f_x \cos \theta_x d\theta_x d\theta_y \\ \Delta \nu_y &= \xi_y \frac{1}{(2\pi)^3 A_y} \int_0^{2\pi} \int_0^{2\pi} f_y \cos \theta_y d\theta_x d\theta_y \end{aligned} \tag{11}$$

and

$$V_{lm} = \frac{\xi_y \sigma_y^2 A_y}{m\beta_y (2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} f_y \cos l\theta_x \sin m\theta_y \sin \theta_y d\theta_x d\theta_y \tag{12}$$

The integration of (11) gives for the tune shifts:

$$\Delta \nu_x \left(A_x, A_y\right) = \frac{\xi_x \sqrt{2}}{\pi A_x} \int_0^{2\pi} F_D\left(\frac{A_x \cos \theta_x}{\sqrt{2}}\right) \cos \theta_x d\theta_x \tag{13}$$

$$\Delta \nu_y \left(A_x, A_y\right) = \xi_y I_0\left(\frac{A_x{}^2}{4}\right)\left[I_0\left(\frac{A_y{}^2}{4}\right) + I_1\left(\frac{A_y{}^2}{4}\right)\right] \exp\left\{-\frac{A_x{}^2 + A_y{}^2}{4}\right\} \tag{14}$$

where I_0 and I_1 are modified Bessel functions of zero and first order. Here we define the dimensionless horizontal and vertical amplitudes of the betatron oscillations as :

$$A_x = \frac{\sqrt{2 I_x \beta_x}}{\sigma_x} \quad ; \quad A_y = \frac{\sqrt{2 I_y \beta_y}}{\sigma_y} \tag{15}$$

For a very flat bunch under the above mentioned assumptions the nonlinearity (8) can be successfully approximated by:

$$\alpha \cong \frac{m^2 \beta_y}{A_y \sigma_y^2} \frac{\partial \Delta \nu_y}{\partial A_y} \tag{16}$$

Substitution of (12) and (16) into (7) gives the resonance width in the amplitude plane:

$$\Delta A_y = 2\sqrt{F_m(A_y) G_l(A_x)} \tag{17}$$

with functions

$$G_l\left(A_x\right) = \frac{I_{l/2}\left(\frac{A_x{}^2}{4}\right)}{I_0\left(\frac{A_x{}^2}{4}\right)} \tag{18}$$

$$F_m\left(A_y\right) = \frac{\pi A_y{}^2}{m^2 - 1} \frac{I_{m/2-1}\left(\frac{A_y{}^2}{4}\right) + \left(1 - \frac{2(m-1)}{A_y{}^2}\right) I_{m/2}\left(\frac{A_y{}^2}{4}\right)}{I_1\left(\frac{A_y{}^2}{4}\right)} \tag{19}$$

Here we consider only the resonance width for the vertical oscillations since the vertical amplitude beat is usually much larger than that of the horizontal oscillations for very flat bunches. Useful integrals used in getting expressions (13)÷(19) are listed in Appendix 1.

More general expressions for the tune shifts and resonance width can be found, for example, in [4] and we also recall them in Appendix 2, reducing the integrals from three-dimensional to one-dimensional. But we must mention that for DAΦNE parameters, where $\sigma_y / \sigma_x = 0.01$, the results of numerical calculations with expressions from Appendix 2 are in a perfect agreement with those given by the analytical expressions (13)÷(19). The great advantage of the analytical formulae is in that they supply us with the explicit dependence of the resonance parameters on the betatron amplitudes and resonance numbers *l* and *m*.

When the bunch length is comparable to the beta-function at the interaction point the synchrotron oscillations give some new effects. A first harmful effect is the betatron tune modulation by the synchrotron motion which leads to creation of synchro-betatron sideband resonances. Two other effects can be considered as useful. Due to the beta-function variation around the IP, the average transverse kick is less than that of a pancake bunch. The modulation of the beta-function with synchrotron oscillations helps in reducing the transverse kick due to averaging over betatron phases.

These effects can be taken into account in the nonlinear resonance width calculation by simply multiplying expression (17) by the so-called "suppression factor" [4]

$$F = \sqrt{\exp\left\{-\frac{1}{2}\left(\frac{m\sigma_s}{2\beta_y}\right)^2\right\} J_n\left(\frac{mc\hat{\tau}}{2\beta_y}\right)} \tag{20}$$

where σ_s is the longitudinal rms bunch length; $\hat{\tau}$ is the amplitude of synchrotron oscillations; J_n is the Bessel function of the n^{th} order. We remind that m and n are the resonance numbers (See Eq.(1)).

It is interesting to note that a horizontal crossing angle can yield an additional suppression of vertical synchrobetatron resonances with the suppression factor [5]:

$$\begin{aligned} F = \sqrt{\frac{2}{\pi}} \int_0^{2\pi} \frac{d\Phi_s}{2\pi} e^{-in\Phi_s} \\ \times \int_{-\infty}^{\infty} du \left(-2\left[u - \frac{\hat{\tau}\cos\Phi_s}{2\sigma_s}\right]^2 - \frac{2\phi^2\sigma_s^2 u^2}{\sigma_x^2} + im\Phi_y(\varsigma u)\right)\sqrt{1+\varsigma^2 u^2} \end{aligned} \tag{21}$$

with $\varsigma = \sigma_s / \beta_y$.

DAΦNE nonlinear beam-beam resonances

Fig. 1 shows the dependence of the tunes on the amplitude of betatron oscillations for the DAΦNE main rings (beam footprint), obtained from equations A2(1)÷A2(3) of Appendix 2. The left corner of the footprint is the machine tune with $\nu_{x0} = 5.18$ and $\nu_{y0} = 6.15$. The right upper corner corresponds to the point with $\nu_x = \nu_{x0} + \Delta\nu_x(0,0)$ and $\nu_y = \nu_{y0} + \Delta\nu_y(0,0)$. The spacings between grid lines are a $0.5\,\sigma_x$ and $1\,\sigma_y$ in the horizontal and vertical directions respectively.

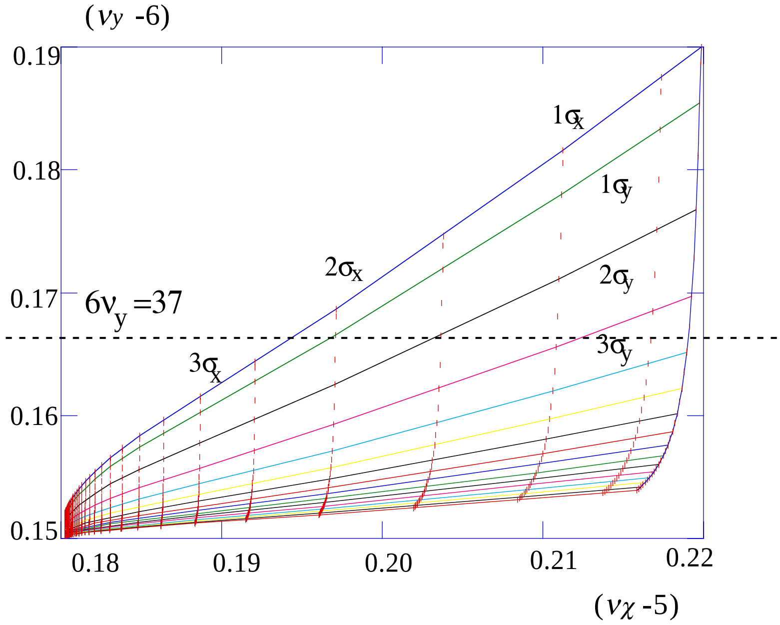

Fig. 1 - DAΦNE footprint.

In the following we consider only purely betatron resonances:

$$l\Big(\nu_x + \Delta\nu_x\big(A_x, A_y\big)\Big) + m\Big(\nu_y + \Delta\nu_y\big(A_x, A_y\big)\Big) = k \qquad (22)$$

taking into account the suppression factor (20). For DAΦNE the horizontal crossing angle does not give substantial suppression of the vertical resonances since the factor $\phi\sigma_s / \sigma_x$ in Eq. (21) is relatively small.

The synchrobetatron resonances with the same numbers l and m usually have the narrowest resonance width. Some comments on them are given in the Discussion.

In particular, we are interested in the sum resonances because resonance streaming occurs only for these resonances. As it can be easily seen, the following sum resonance lines cross the beam footprint:

$$6\nu_y = 37; \quad 10\nu_x = 52; \quad 4\nu_y + 2\nu_x = 35; \quad 6\nu_y + 4\nu_x = 58; \quad 4\nu_y + 6\nu_x = 56$$

and others of higher orders. The resonance $6\nu_y = 37$ shown in Fig. 1 passes near the bunch core and is expected to contribute to the vertical bunch blow up.

Although the resonances $6\nu_y + 4\nu_x = 58$ and $4\nu_y + 6\nu_x = 56$ pass even closer to the bunch core, their resonance width (17) drops by the factor $\sqrt{G_l(A_x)}$ which is particularly small for small amplitudes A_x. The resonance $4\nu_y + 2\nu_x = 35$ touches the bunch tail contributing to the appearance of non Gaussian tails in the vertical direction.

The dangerous resonance $6\nu_y = 37$ is shown in Fig.2 on the amplitude plane.

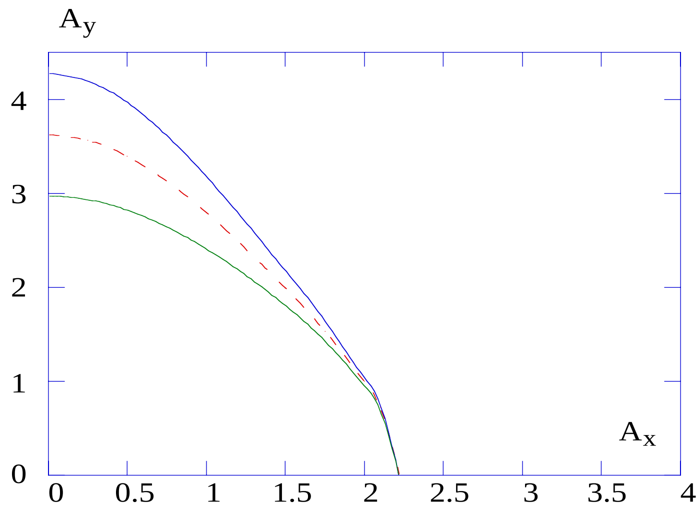

Fig. 2 - Resonance $6\nu_y = 37$ *on the amplitude plane.*

We can see that the resonance passes in the vicinity of the beam core. Due to the streaming along the resonance line (dotted line) vertical amplitudes of particles staying within the resonance increase. For example, there is a high probability that a particle with $A_x = 1.5$; $A_y = 2$ will change its coordinates to $A_x = 0$; $A_y = 3.5$. We should mention also that the resonance streaming together with the phase convection mechanism change the particle distribution not only inside the resonance region (between the solid lines of Fig. 2), but also below the lower resonance line. All this can lead to substantial vertical bunch blow-up.

Discussion

1. The present working point with $\nu_{x0} = 5.18$; $\nu_{y0} = 6.15$ is not probably the best choice because of the presence of the strong beam-beam resonance $6\nu_y = 37$. More precise information about the vertical beam blow-up could be given by numerical simulations taking into account noise and radiation damping.

2. The strength of the vertical beam-beam resonances can be reduced by increasing the bunch length and even increasing the horizontal crossing angle (See (20) and (21)). It should not be forgotten that these remedies yield a geometrical decrease of the luminosity. Nevertheless, some compromise could be reached. In particular, it is believed that the phase averaging mechanism contributed to good CESR performance with $\sigma_s / \beta_y \sim 1.1$ [7].

3. The best way to substantially reduce the strength of the beam-beam nonlinear resonances is to move the working point into the region where there are only higher order resonances, i.e. closer to the integer.

4. The synchro-betatron resonances can give additional troubles, adding new resonances of odd low order, like $5\nu_y \pm \nu_s$. Moreover, the synchrobetatron sidebands separated by ν_s can overlap and excite chaotic motion. However, the DAΦNE synchrotron tune is very small ($\nu_s = 0.0077$ or even smaller in the bunch lengthening regime) and an adiabatic regime with stable motion instead of the chaotic one can take place. The numerical simulation also could give right answers to the question.

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APPENDIX 1 Useful integrals

$$\int_0^{2\pi} \exp\left\{-\frac{A_x^2 \cos^2\theta_x}{2}\right\} \cos l\theta_x d\theta_x = 2\pi I_{l/2}\left(\frac{A_x^2}{4}\right) \exp\left\{-\frac{A_x^2}{4}\right\} (-1)^{l/2} \qquad \text{A1(1)}$$

$$\int_0^{2\pi} \exp\left\{-\frac{A_x^2 \cos^2\theta_x}{2}\right\} \cos^4\theta_x d\theta_x = \pi \exp\left\{-\frac{A_x^2}{4}\right\} \left[\frac{3}{4} I_0\left(\frac{A_x^2}{4}\right) - I_1\left(\frac{A_x^2}{4}\right) + \frac{1}{4} I_2\left(\frac{A_x^2}{4}\right)\right] \qquad \text{A1(2)}$$

$$\int_0^{2\pi} \left[\int \Phi\left(\frac{A_y \cos\theta_y}{\sqrt{2}}\right) dA_y \cos\theta_y\right] \cos m\theta_y d\theta_y = \frac{A_y}{m} \int_0^{2\pi} \Phi\left(\frac{A_y \cos\theta_y}{\sqrt{2}}\right) \sin m\theta_y \sin\theta_y d\theta_y \qquad \text{A1(3)}$$

$$\int_0^{2\pi} \Phi\left(\frac{A_y \cos\theta_y}{\sqrt{2}}\right) \sin m\theta_y \sin\theta_y d\theta_y =$$

$$\sqrt{2\pi}\, \frac{m}{m^2-1} A_y \exp\left\{-\frac{A_y^2}{4}\right\} \left[I_{m/2-1}\left(\frac{A_y^2}{4}\right) + \left(1 - \frac{2(m-1)}{A_y^2}\right) I_{m/2}\left(\frac{A_y^2}{4}\right)\right] \qquad \text{A1(4)}$$

In the integrals A1(3) and A1(4) $\Phi(z)$ is the error function (probability integral) $\Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp\left\{-t^2\right\} dt$.

APPENDIX 2. Tune shifts and resonance width.

1. Tune shift in the y-direction [4]:

$$\nu_y = \nu_{y0} + \frac{2Nr_e\beta_y}{\gamma(2\pi)^3}\int_0^{2\pi} d\theta_x \int_0^{2\pi} d\theta_y \cos^2\theta_y \int_0^{\infty} \frac{dq}{\sqrt{\left(2\sigma_x^2+q\right)\left(2\sigma_y^2+q\right)^3}} \qquad \text{A2(1)}$$

$$\times \exp\left\{-\left[\frac{2\beta_x I_x \cos^2\theta_x}{2\sigma_x^2+q} + \frac{2\beta_y I_y \cos^2\theta_y}{2\sigma_y^2+q}\right]\right\}$$

We can somewhat simplify this expression by reducing the three-dimensional integral to one dimensional:

$$\nu_y = \nu_{y0} + \frac{1}{2}\xi_y\left(\frac{\sigma_y}{\sigma_x}\right)\left(1+\frac{\sigma_y}{\sigma_x}\right)\int_0^{\infty} \frac{I_0(a)\left[I_0(b)-I_1(b)\right]\exp\{-(a+b)\}}{\sqrt{(1+x)\left(\left(\sigma_y/\sigma_x\right)^2+x\right)^3}} \qquad \text{A2(2)}$$

where

$$a = \frac{A_x^2}{4(1+x)}; \quad b = \frac{A_x^2\left(\sigma_y/\sigma_x\right)^2}{4(\left(\sigma_y/\sigma_x\right)^2+x)} \qquad \text{A2(3)}$$

with A_x; A_y being dimensionless betatron amplitudes defined by (15). The corresponding expression for the horizontal plane can be obtained by interchanging indexes x and y.

2. The full resonance width is [4]:

$$\Delta K_1 = 4\sqrt{\left|\frac{2F_{lmn}^R}{\Lambda_{lm}}\right|} \qquad \text{A2(4)}$$

where the action K_1 is related to the betatron actions $I_x; I_y$ by the relation:

$$\frac{I_x}{l} + \frac{I_y}{m} = 2K_1 \qquad \text{A2(5)}$$

The coefficient F_{lmn} is defined by:

$$F_{lmn}\left(I_x, I_y, \hat{\tau}\right) = T_{lm}\left(I_x, I_y\right)\exp\left\{-\frac{1}{2}\left(\frac{m\sigma_s}{2\beta_y}\right)^2\right\} J_n\left(\frac{mc\hat{\tau}}{2\beta_y}\right) \qquad \text{A2(6)}$$

with

$$T_{lm} = \frac{1}{(2\pi)^2}\int_0^{2\pi} d\theta_x e^{-ilx}\int_0^{2\pi} d\theta_y e^{-imy}\int_0^{\infty}\frac{dq}{\sqrt{\left(2\sigma_x{}^2+q\right)\left(2\sigma_y{}^2+q\right)}}$$

$$\times\exp\left\{-\left(\frac{2\beta_x I_x\cos^2\theta_x}{2\sigma_x{}^2+q}+\frac{2\beta_y I_y\cos^2\theta_y}{2\sigma_y{}^2+q}\right)\right\} \qquad \text{A2(7)}$$

The nonlinearity Λ_{lm} in this case is given by:

$$\Lambda_{lm} = \left[l^2\frac{\partial^2 T_{00}}{\partial I_x^2}+m^2\frac{\partial^2 T_{00}}{\partial I_y^2}+2lm\frac{\partial^2 T_{00}}{\partial I_x \partial I_y}\right]_R \qquad \text{A2(8)}$$

The index "R" in A2(4) and A2(8) indicates that the values are calculated for the resonance conditions. The expression for T_{lm} can be simplified to:

$$\left|T_{lm}\right| = \int_0^{\infty}\frac{I_{l/2}(a)I_{m/2}(b)\exp\{-(a+b)\}}{\sqrt{(1+x)\left(\left(\sigma_y/\sigma_x\right)^2+x\right)}}dx \qquad \text{A2(9)}$$

We give also the expression for the second term of Λ_{lm} which plays a dominant role in the vertical amplitude beat calculations:

$$\frac{\partial^2 T_{00}}{\partial I_y^2} = \frac{\beta_y^2}{2\sigma_x^4}\int_0^{\infty}\frac{I_0(a)\left[0.75 I_0(b)-I_1(b)+0.25 I_2(b)\right]\exp\{-(a+b)\}}{\sqrt{(1+x)\left(\left(\sigma_y/\sigma_x\right)^2+x\right)^5}}dx \qquad \text{A2(10)}$$

The parameters a and b are as given by A2(3).